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A NOTE ON THE gl CONSTANT OF $E \check{\otimes} F$

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ABSTRACT

Let X and Y be Banach spaces. TFAE (1) X and Y do not contain subspaces uniformly isomorphic to l_∞^n's. (2) The local unconditional structure constant of the space of bounded operators $L(X_k^*, Y_k)$ tends to infinity for every increasing sequence $\{X_k\}_{k=1}^\infty$ and $\{Y_k\}_{k=1}^\infty$ of finite-dimensional subspaces of X and Y respectively.

Given Banach spaces E and F, let $E \check{\otimes} F$ denote the closure of the finite rank tensors $t = \sum_{i=1}^n x_i \check{\otimes} y_i \in E \check{\otimes} F$ in the norm $|t|_\vee = \sup\{|\Sigma\langle x_i, x^*\rangle\langle y_i, y^*\rangle|; \|x^*\| = \|y^*\| = 1\}$. Let (Π_1, π_1) be the normed ideal of 1-absolutely summing operators, and (Γ_1, γ_1) the normed ideal of L_1-factorizable operators. It is well known that if $u \in L(E, F)$ and $v \in L(F, E)$ have finite ranks, then $\operatorname{trace}(uv) \leqq \gamma_1(u)\pi_1(v^*)$ ([7]). The gl constant of a Banach space X is defined as $\mathrm{gl}(X) = \sup\{\gamma_1(T); \pi_1(T) = 1, T \in \Pi_1(X, l_2)\}$, and it is also well known that $\mathrm{gl}(X) \leqq \chi_u(X)$ (= the local unconditional structure constant of X) [6].

THEOREM. *Let X and Y be Banach spaces. Then $\mathrm{gl}(X_k \check{\otimes} Y_k) \to_{k\to\infty} \infty$ for every increasing sequence $\{X_k\}_{k=1}^\infty$ and $\{Y_k\}_{k=1}^\infty$ of finite-dimensional subspaces of X and Y respectively, if, and only if, X and Y do not contain subspaces uniformly isomorphic to l_∞^n's.*

We shall in fact obtain a quantitative version of the theorem which is almost exact. The "only if" part of the theorem is very simple, because suppose X has subspaces X_k uniformly isomorphic to l_∞^k's. By Dvoretzky's theorem, Y contains subspaces Y_k uniformly isomorphic to l_2^k's, and since $l_\infty^k \check{\otimes} l_2^k$ has a (natural) monotone unconditional basis, therefore the sequence $\{\mathrm{gl}(X_k \check{\otimes} Y_k)\}_{k=1}^\infty$ is bounded.

For the "if" part of the proof we need some definitions and other results. Let

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$\{g_i(w)\}_{i=1}^n$ be centered independent normalized Gaussian variables on a probability space (Ω, P) and X be a Banach space. Following [3] we define on $L_2(X, \Omega, P)$ the projection G_n by $G_n(f) = \sum_{i=1}^n g_i \int_\Omega f(w) g_i(w) dP(w)$. Clearly the range of G_n can be naturally and isometrically identified with the linear space $L(l_2^n, X)$ equipped with the norm

$$l(T) = \left(\int_\Omega \left\| \sum_{i=1}^n g_i(w) T(e_i) \right\|^2 dP(w) \right)^{1/2} \qquad (T \in L(l_2^n, X)),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis for l_2^n. The norm dual to l, l^*, is defined for operators $S \in L(X, l_2^n)$ by $l^*(S) = \sup\{\operatorname{trace}(ST);\ T \in L(l_2^n, X),\ l(T) = 1\}$. It is known that if $T \in L(l_2^n, X)$, then $l^*(T^*) \leqq l(T) \leqq \gamma_n(X) l^*(T^*)$, where $\gamma_n(X) = \|G_n\|$; if in addition $\dim X = n$, then $\gamma_n(X) \leqq c \log(n+1)$, where c is an absolute constant [9] (see [1] for related facts and references).

The Gaussian type p (cotype q) constant of a Banach space X on n vectors is denoted by $\alpha_n^{(p)}(X)$ ($\beta_n^{(q)}(X)$), and defined to be the least C such that

$$\left(\int_\Omega \left\| \sum_{i=1}^n g_i x_i \right\|^2 \right)^{1/2} \leqq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$

$$\left(\left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leqq C \left(\int_\Omega \left\| \sum_{i=1}^n g_i x_i \right\|^2 \right)^{1/2} \right)$$

for all $x_1, x_2, \cdots, x_n$ in X. We shall use the inequalities $\alpha_n^{(2)}(X^*) \leqq \gamma_n(X) \beta_n^{(2)}(X)$ and $\beta_n^{(2)}(X) \leqq n^{1/2-1/q} \beta_n^{(q)}(X)$. Moreover, if $\dim X = N$, then $\alpha_n^{(2)}(X) \leqq 2\alpha_N^{(2)}(X)$ for all $n \geqq 1$ ([10]).

Proof of the "If" Part of the Theorem. We shall first prove the following two inequalities which may be of independent interest:

(I) If E and F have finite-dimensions and $A \in L(l_2^n, E)$, $B \in L(l_2^n, F)$, then $l^*(A^*) l^*(B^*) \leqq c \operatorname{gl}(E \hat{\otimes} F)[\|A\| l(B) + \|B\| l(A)]$.

(II) If $\dim E = n$, $\dim F = m$, then

$$1 \leqq c \operatorname{gl}(E \hat{\otimes} F)[n^{-1/2} \alpha_n^{(2)}(E^*)(\log(m+1)) + m^{-1/2} \alpha_m^{(2)}(F^*)(\log(n+1))]$$

(c always denotes an absolute constant).

(I) An easy consequence of proposition 9(1) [5] states that if $T \in L(X, l_2^n)$, then $\pi_1(T) \leqq c\sqrt{n} (\int_S \|T^*(x)\|^2 dm(x))^{1/2} = cl(T^*)$, where dm denotes the normalized rotation invariant measure on the unit sphere S_n of l_2^n. Let now $C \in L(E, l_2^n)$ and $D \in L(F, l_2^n)$ be arbitrary, and consider the maps

$$E \check{\otimes} F \xrightarrow{C\otimes D} \mathrm{HS}(l_2^n) \xrightarrow{A\otimes B} E \check{\otimes} F,$$

where $\mathrm{HS}(l_2^n)$ $(= l_2^{n^2})$ is the space of Hilbert Schmidt operators on l_2^n. By proposition 5.2 [4], $\pi_1(C \otimes D) \leqq \pi_1(C)\pi_1(D)$, hence

$$\gamma_1(C \otimes D) \leqq \mathrm{gl}(E \check{\otimes} F)\pi_1(C \otimes D) \leqq \mathrm{gl}(E \check{\otimes} F)\pi_1(C)\pi_1(D)$$
$$\leqq c^2 \mathrm{gl}(E \check{\otimes} F) l(C^*) l(D^*),$$

therefore, combining inequalities we obtain

$$\mathrm{trace}(AC)\,\mathrm{trace}(BD) = \mathrm{trace}((A \otimes B)(C \otimes D)) \leqq \gamma_1(C \otimes D)\pi_1((A \otimes B)^*)$$
$$\leqq c^2 \mathrm{gl}(E \check{\otimes} F) l(C^*) l(D^*) c l(A \otimes B).$$

Now, by [2] $l(A \otimes B) \leqq c[\|A\| l(B) + \|B\| l(A)]$, and this, together with the definition of l^* establishes (I).

(II) Using lemma 4.11 [1], there exist $A \in L(l_2^n, E)$, $B \in L(l_2^m, F)$, satisfying

$$\|A\| = \|B\| = 1, \quad l(A^{*-1}) \leqq 2\sqrt{n}\,\alpha_n^{(2)}(E^*) \quad \text{and} \quad l(B^{*-1}) \leqq 2\sqrt{m}\,\alpha_m^{(2)}(F^*).$$

Therefore, $\|A\|/l^*(A^*) = 1/l^*(A^*) \leqq n^{-1} l(A^{*-1}) \leqq 2n^{-1/2}\alpha_n^{(2)}(E^*)$ and $l(A)/l^*(A^*) \leqq c\log(n+1)$ and similar estimates hold for the operator B as well. Now obviously (I) implies (II).

By [8] X and Y have some cotype $q < \infty$, i.e. $\sup_n \beta_n^{(q)}(X)$ and $\sup \beta_n^{(q)}(Y) \leqq c_1 < \infty$. Hence, if $\dim X_k = n_k$

$$\alpha_{n_k}^{(2)}(X_k^*) \leqq \gamma_{n_k}(X_k)\beta_{n_k}^{(2)}(X_k) \leqq c(\log(n_k+1)) c n_k^{1/2-1/q}\beta_{n_k}^{(q)}(X_k)$$
$$\leqq c^2 c_1(\log(n_k+1)) n_k^{1/2-1/q}$$

with a similar estimate holding for the space Y_k of dimension m_k. Applying these inequalities in (II) we obtain

$$1 \leqq c^3 c_1 \mathrm{gl}(X_k \check{\otimes} Y_k)(\log(n_k+1)\log(m_k+1))(n_k^{-1/q} + m_k^{-1/q}),$$

that is $\mathrm{gl}(X_k \check{\otimes} Y_k) \to_{k\to\infty} \infty$. □

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